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Periodic orbits of Hamiltonian systems and symplectic reduction

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Abstract. The notion of relative periodic orbits for Hamiltonian systems with symmetry is discussed and a correspondence between periodic orbits of reduced and unreduced Hamiltonian systems is established. Variational principles with symmetries are studied from the point of view of symplectic reduction of the space of loops, leading to a characterization of reduced periodic orbits by means of the critical subsets of an action functional restricted to a submanifold of the loop space of the unreduced manifold. Finally, as an application, it is shown that if the symplectic form ω has finite integral rank, then the periodic orbits of a Hamiltonian system on the symplectic manifold (M, ω) admit a variational characterization.

1. Introduction

One of the first domains of research in the theory of dynamical Hamiltonian systems has been the study of their equilibrium points and periodic orbits. In the presence of symmetries such study leads to the analysis of relative equilibrium points [14] (see [13] and references therein for an updated description of the problem). We shall extend some of the ideas involved in such an analysis to periodic solutions of Hamiltonian systems with symmetries. In this setting the notion of relative periodic orbit arises naturally and leads to the existence of a one-to-one correspondence between periodic orbits of reduced Hamiltonian systems, and certain families of periodic orbits in the unreduced manifold.

Periodic orbits with period $\tau \in \mathbb{R}$ of a Hamiltonian dynamical system defined by the Hamiltonian H_t in an exact symplectic manifold (M, ω), $\omega = -\mathrm{d}\theta$, are characterized (Hamilton's principle) as critical points of the action functional

$$\mathcal{A}_H(u) = \frac{1}{\tau}\int_u \theta - \frac{1}{\tau}\int_0^\tau H_t(u(t))\,\mathrm{d}t \tag{1.1}$$

defined on the space of smooth free loops on M:

$$\mathcal{L}_\tau(M) = \{u \in C^\infty(\mathbb{R}, M) \mid u(t+\tau) = u(t), \forall\ t \in \mathbb{R}\}. \tag{1.2}$$

In arbitrary symplectic manifolds (M, ω), the action functional is not well defined and one only has a map $\mathcal{A}_H$ defined on contractible loops and taking values in the quotient $\mathbb{R}/\Gamma$, where Γ denotes the period group of ω.

Many symplectic manifolds arise as the Marsden–Weinstein reduction of exact symplectic manifolds. In this situation it is possible to establish a correspondence between periodic orbits of the reduced system and certain critical sets of an action functional on the loop space of the unreduced manifold.

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Variational principles for reduced dynamical systems have been partially studied in [2], [3] and references therein. In that approach the main idea was to use Lin constraints and Clebsch variables to obtain a variational description of reduced Lagrangian systems. In this paper we shall use instead a direct approach showing that the free loop space of the reduced phase space can be obtained by symplectic reduction of the free loop space of the original symplectic manifold. Some of these ideas have been originated by Fortune's proof of Arnold's conjecture for $\mathbb{C}P^n$ [5, 4].

The paper is organized as follows. Section 2 introduces the notion of relative periodic orbit, which leads to a connection between periodic orbits of reduced and unreduced Hamiltonian systems. Section 3 studies the variational characterization of periodic orbits in manifolds obtained as the Marsden–Weinstein reduction of exact symplectic manifolds. Section 4 is devoted to showing how the previous ideas can be applied to symplectic toric actions Hamiltonian systems defined on symplectic manifolds (M, ω) with ω of finite integral rank.

2. Periodic orbits in reduced Hamiltonian systems

Let G be a connected Lie group acting smoothly and symplectically on a symplectic manifold (M, ω). The action $\Phi: G \times M \to M$ will be denoted either by $(g, m) \mapsto \Phi_g(m)$ or simply by $(g, m) \mapsto g \cdot m$. Let us assume that the action admits an Ad^*-equivariant momentum map $J : M \to \mathfrak{g}^*$, where $\mathfrak{g}^*$ denotes the dual of the Lie algebra $\mathfrak{g}$ of G. This means that the infinitesimal generators ξ_M of the G-action on M defined by the elements $\xi \in \mathfrak{g}$ are Hamiltonian:

$$i_{\xi_M}\omega = \mathrm{d}J_\xi \qquad \forall\ \xi \in \mathfrak{g} \tag{2.1}$$

with Hamiltonians $J_\xi = \langle \xi, J \rangle$, and

$$J(g \cdot m) = \mathrm{Ad}^*_{g^{-1}}\, J(m) \qquad \forall\ g \in G \quad \forall\ m \in M. \tag{2.2}$$

Let $\mu \in \mathfrak{g}^*$ be a regular value of J. Due to equation (2.2) there is an induced smooth action of the isotropy subgroup G_μ of μ on the submanifold $J^{-1}(\mu)$. If the quotient space $M_\mu = J^{-1}(\mu)/G_\mu$ is a manifold and the projection map $\pi_\mu: J^{-1}(\mu) \to M_\mu$ is a submersion, then there is an induced symplectic form ω_μ on M_μ, defined as the unique symplectic form satisfying

$$\pi_\mu^*\omega_\mu = i_\mu^*\omega \tag{2.3}$$

where $i_\mu: J^{-1}(\mu) \to M$ is the inclusion. The symplectic manifold (M_μ, ω_μ) is called the Marsden–Weinstein reduction of (M, ω) relative to μ. If the action of G_μ is proper and free, then $J^{-1}(\mu) \xrightarrow{\pi_\mu} M_\mu$ is a principal fibre bundle with structural group G_μ.

We shall assume in what follows that G_μ is compact, connected, and acts freely on $J^{-1}(\mu)$, so that the conditions above are automatically satisfied.

Consider now a time-dependent Hamiltonian $H: M \times \mathbb{R} \to \mathbb{R}$ on M such that each $H_t = H(\cdot, t)$ is G-invariant, and denote by X_{H_t} its associated time-dependent Hamiltonian vector field, i.e.

$$i(X_{H_t})\omega = \mathrm{d}H_t. \tag{2.4}$$

The integral curves of X_{H_t} will be the solutions of

$$\frac{\mathrm{d}}{\mathrm{d}t}u(t) = X_{H_t}(u(t)) \tag{2.5}$$

and we shall denote by σ_m the solution with initial value $u(0) = m \in M$. The restriction to $J^{-1}(\mu)$ of the Hamiltonian vector field X_{H_t} is tangent to $J^{-1}(\mu)$ and G_μ-invariant. The projection $T\pi_\mu(X_{H_t})$ is a well defined Hamiltonian vector field on M_μ, called the reduced Hamiltonian vector field, whose associated Hamiltonian h_t satisfies $h_t \circ \pi_\mu = H_t|_{J^{-1}(\mu)}$.

If we denote by σ_{μ,m_μ} the solution of

$$\frac{\mathrm{d}}{\mathrm{d}t}u(t) = X_{h_t}(u(t)) \tag{2.6}$$

with initial value $u(0) = m_\mu \in M_\mu$, then we have

$$\pi_\mu \circ \sigma_m = \sigma_{\mu,\pi_\mu(m)} \qquad \forall\, m \in J^{-1}(\mu). \tag{2.7}$$

The most interesting solutions of Hamiltonian systems with symmetry are their relative equilibria. Recall that a point $m \in J^{-1}(\mu)$ is called a relative equilibrium if $\pi_\mu(m)$ is an equilibrium of the reduced Hamiltonian system defined by h_t. In terms of the flow of X_{H_t}, a point $m \in J^{-1}(\mu)$ is a relative equilibrium if and only if there exists a curve $t \mapsto g(t)$ in G_μ such that the solution of (2.5) with initial value m is of the form $\sigma_m(t) = g(t) \cdot m$. If H is time-independent, then the curve above must be of the form $g(t) = \exp t\xi$ where exp denotes the exponential map of G_μ.

A useful characterization of relative equilibria is given by the following proposition (see [1]):

Proposition 1. A point $m \in J^{-1}(\mu)$ is a relative equilibrium if and only if m is a critical point of $H_t \times J : M \to \mathbb{R} \times \mathfrak{g}^*$ for each $t \in \mathbb{R}$.

Note that, by the Lagrange multipliers theorem, the critical points of $H_t \times J$ lying on $J^{-1}(\mu)$ are exactly the critical points of the restriction $H_t\big|_{J^{-1}(\mu)}$.

After equilibrium points, the most interesting solutions of Hamilton's equations are periodic orbits. We shall assume that the Hamiltonian H is 1-periodic in time, i.e. $H(m, t+1) = H(m, t)$. By periodic orbits we shall mean periodic solutions of Hamilton's equations with integer period.

Note that, because H is 1-periodic, the relation

$$\sigma_m(t+\tau) = \sigma_{\sigma_m(\tau)}(t) \qquad \forall\, t \in \mathbb{R} \tag{2.8}$$

holds for each $\tau \in \mathbb{N}$ and a solution u of (2.5) will be periodic, with integer period τ, if and only if $u(0) = u(\tau)$.

In the presence of symmetries, the reduced Hamiltonian h of H will also be 1-periodic, and we have the following natural generalization of the notion of relative equilibrium.

Definition 1. An integral curve u of the Hamiltonian vector field X_{H_t} is called a relative periodic orbit if it projects on a periodic orbit of the reduced Hamiltonian system defined by h_t.

It is easily seen that an integral curve u will be a relative periodic orbit if and only if $J(u(0)) = \mu$ (which implies $J(u(t)) = \mu, \quad \forall\, t \in \mathbb{R}$) and there exist $\tau \in \mathbb{N}$, $g \in G_\mu$ such that

$$u(0) = g \cdot u(\tau). \tag{2.9}$$

The notion of relative periodic orbit also arises starting from periodic solutions of Hamiltonian systems on reduced phase spaces. That is, if h_t is a 1-periodic Hamiltonian on M_μ, we are looking now for periodic solutions with integer period of (2.6). We shall denote by $\mathcal{P}_h^\tau$ the set of all τ-periodic solutions, i.e.

$$\mathcal{P}_h^\tau = \{u \in \mathcal{L}_\tau(M_\mu) \mid u \text{ satisfies equation (2.6)}\}. \tag{2.10}$$

Now we shall assume that there exists a time-dependent 1-periodic Hamiltonian H on M such that each H_t is a G-invariant extension to M of the pull-back $\pi_\mu^* h_t$. Then X_{h_t} is the reduced Hamiltonian vector field corresponding to X_{H_t}.

If the solution σ_{μ,m_μ} of (2.6) with initial value m_μ belongs to $\mathcal{P}_h^\tau$ and $m \in \pi_\mu^{-1}(m_\mu)$, then the solution σ_m of (2.5) with initial value m will satisfy $\pi_\mu(\sigma_m(0)) = \pi_\mu(\sigma_m(\tau))$ and hence there will exist $g_m \in G_\mu$ such that

$$m = \sigma_m(0) = g_m \cdot \sigma_m(\tau). \tag{2.11}$$

In other words, σ_m will be a relative periodic orbit of the unreduced system. The element g_m will be called the holonomy of the curve σ_m.

For each $g \in G_\mu$, $\Phi_g \circ \sigma_m = \sigma_{g\cdot m}$ will also be a relative periodic orbit of X_{H_t} projecting on σ_{μ,m_μ}, whose holonomy is just $g g_m g^{-1}$. In fact, there is a one-to-one correspondence between periodic orbits σ_{μ,m_μ} of X_{h_t} and families of relative periodic orbits of X_{H_t} with initial values at the points of the G_μ-orbit $\pi_\mu^{-1}(m_\mu)$. There is also a conjugation class $\mathcal{C}_\sigma \subset G_\mu$ made of the holonomies $g g_m g^{-1}$ associated with a periodic orbit σ_{μ,m_μ}.

The variational characterization in section 3 of periodic orbits in reduced systems requires to establish a correspondence between them and periodic orbits of unreduced Hamiltonian systems. This can be done if one allows the unreduced Hamiltonian to vary in the set $\{H_t + J_\xi \mid \xi \in \mathfrak{g}_\mu\}$. (Note that the reduced Hamiltonian vector field of $X_{H_t+J_\xi} = X_{H_t} + \xi_M$ is again X_{h_t}.)

More precisely, let us consider the equation

$$\frac{\mathrm{d}}{\mathrm{d}t} u(t) = X_{H_t+J_\xi}(u(t)) \tag{2.12}$$

and define

$$\widehat{\mathcal{P}}_H^\tau = \{u^\xi \in \mathcal{L}_\tau(J^{-1}(\mu)) \mid u^\xi \text{ solves equation (2.12), for some } \xi \in \mathfrak{g}_\mu\}. \tag{2.13}$$

The projection π_μ defines a map $\Lambda(\pi_\mu): \mathcal{L}_\tau(J^{-1}(\mu)) \to \mathcal{L}_\tau(M_\mu)$ by $\Lambda(\pi_\mu)(u) = \pi_\mu \circ u$, which in turn induces a map $\widehat{\Lambda}(\pi_\mu): \widehat{\mathcal{P}}_H^\tau \to \mathcal{P}_h^\tau$. Our aim is to describe the set $\widehat{\Lambda}(\pi_\mu)^{-1}(u_\mu)$ for a given $u_\mu \in \mathcal{P}_h^\tau$.

If σ_m^ξ denotes the solution of (2.12) with initial value $m \in M$, then it is not difficult to check that

$$\sigma_m^\xi(t) = (\exp t\xi) \cdot \sigma_m(t). \tag{2.14}$$

Moreover, if $\sigma_m^\xi \in \widehat{\mathcal{P}}_H^\tau$, then σ_m must be a relative periodic orbit with holonomy $g_m = \exp(\tau\xi)$. The curves in $\widehat{\Lambda}(\pi_\mu)^{-1}(u_\mu)$ for a given $u_\mu \in \mathcal{P}_h^\tau$ are thus determined by the two conditions

$$\pi_\mu(m) = u_\mu(0) \qquad \text{and} \qquad \exp(\tau\xi) = g_m \tag{2.15}$$

where $m = u^\xi(0)$. Since G_μ is assumed to be compact and connected, the exponential map is surjective and so is $\widehat{\Lambda}(\pi_\mu)$.

Now, it is easily seen that, for any $g \in G_\mu$,

$$\Phi_g \circ \sigma_m^\xi = \sigma_{g\cdot m}^{\mathrm{Ad}_g \xi}. \tag{2.16}$$

The relation $\exp(\mathrm{Ad}_g\xi) = g(\exp\xi)g^{-1}$ establishes a bijection between $\exp^{-1}(g g_m g^{-1})$ and $\exp^{-1}(g_m)$, which leads to a bijection between the subsets of $\widehat{\Lambda}(\pi_\mu)^{-1}(u_\mu)$ defined by the initial values m and $g \cdot m$, respectively.

Summing up, we can state:

Proposition 2. The map $\widehat{\Lambda}(\pi_\mu)\colon \widehat{\mathcal{P}}^\tau_H \to \mathcal{P}^\tau_h$ is surjective. The set $\widehat{\Lambda}(\pi_\mu)^{-1}(u_\mu)$ is given, for any $m \in \pi_\mu^{-1}(u_\mu(0))$, by

$$\widehat{\Lambda}(\pi_\mu)^{-1}(u_\mu) = \{\Phi_g \circ \sigma^\xi_m \mid \exp(\tau\xi) = g_m,\, g \in G_\mu\}. \tag{2.17}$$

Moreover, there is a bijection $\widehat{\Lambda}(\pi_\mu)^{-1}(u_\mu) \to (G_\mu \cdot m) \times \exp^{-1}(g_m)$ given by $\Phi_g \circ \sigma^\xi_m \mapsto (g \cdot m, \tau\xi)$.

Note that if G is an Abelian group, then $G_\mu = G$ and the exponential map $\exp\colon G \to \mathfrak{g}$ is a local diffeomorphism at every point. For each g_m, the preimage $\exp^{-1}(g_m)$ is a discrete set bijective to $\exp^{-1}(e)$, where e is the identity of G, and $\widehat{\Lambda}(\pi_\mu)^{-1}(u_\mu)$ is a discrete set of G-orbits in the loop space.

There is a close relation between periodic orbits of time-dependent Hamiltonians and periodic points of certain symplectic diffeomorphisms. A symplectic diffeomorphism φ of (M, ω) is said to be exact if it can be obtained by integrating a time-dependent 1-periodic Hamiltonian vector field. More precisely, φ will be exact if there exists a smooth 1-periodic Hamiltonian $H : \mathbb{R} \times M \to \mathbb{R}$ such that, defining φ_t by

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi_t = X_{H_t} \circ \varphi_t \qquad \varphi_0 = id_M \tag{2.18}$$

one has $\varphi = \varphi_1$. A point $m \in M$ is called a periodic point with period $\tau \in \mathbb{N}$ of φ if $\varphi^\tau(m) = m$. In particular, the periodic points of φ with period equal to 1 are just its fixed points.

The relation $\varphi_t(m) = \sigma_m(t)$ gives a one-to-one correspondence between periodic orbits of X_{H_t} and periodic points of the exact symplectomorphism φ_1. In fact, since H is 1-periodic in time, we have

$$\varphi_{t+1} = \varphi_t \circ \varphi_1 \qquad \forall\, t \in \mathbb{R}. \tag{2.19}$$

In particular, if $\tau \in \mathbb{N}$ then $\varphi_\tau = \varphi_1^\tau$, and m is a τ-periodic point of φ_1 if and only if it is the initial value of a τ-periodic orbit of the Hamiltonian system defined by H_t.

In the presence of symmetries, we can consider not only periodic points, but also relative periodic points. A point $m \in M$ will be a relative periodic point with period $\tau \in \mathbb{N}$ of a symplectic diffeomorphism φ of M if there exists $g_m \in G$ such that $\varphi^\tau(m) = g_m \cdot m$. If φ is equivariant, i.e.

$$\varphi \circ \Phi_g = \Phi_g \circ \varphi \qquad \forall\, g \in G \tag{2.20}$$

then its periodic points appear in whole G-orbits, because

$$\varphi^\tau(m) = g_m \cdot m \;\Rightarrow\; \varphi^\tau(g \cdot m) = g \cdot \varphi^\tau(m) = (g g_m g^{-1}) \cdot (g \cdot m). \tag{2.21}$$

As before, the notion of relative periodic point arises from the study of periodic points of exact symplectomorphisms on reduced symplectic manifolds. Let φ_μ be an exact symplectomorphism of M_μ induced by the 1-periodic Hamiltonian h on M_μ, and assume that there exists a 1-periodic Hamiltonian H on M such that each H_t is a G-invariant extension to M of the pull-back $\pi_\mu^* h_t$. Then, the exact symplectomorphism φ induced by H is equivariant, leaves $J^{-1}(\mu)$ invariant and satisfies

$$\pi_\mu \circ \varphi^\tau|_{J^{-1}(\mu)} = \varphi_\mu^\tau \circ \pi_\mu \qquad \forall\, \tau \in \mathbb{N}. \tag{2.22}$$

The previous relation gives a one-to-one correspondence between τ-periodic points of φ_μ and G_μ-orbits in $J^{-1}(\mu)$ of relative τ-periodic points of φ.

Since the τ-periodic points of φ_μ are in one-to-one correspondence with τ-periodic orbits of X_{h_t}, the variational characterization of the latter in section 3 will serve as a variational characterization of the former.

3. Hamilton's principle and symplectic reduction

In this section we shall develop a variational characterization of periodic orbits of reduced Hamiltonian systems (or, equivalently, of periodic points of reduced exact symplectic diffeomorphisms). Such a problem requires understanding the relation between symplectic reduction and Hamilton's principle. We shall provide a systematic study of variational principles with symmetries completing the results obtained in [2].

We shall assume in what follows that the symplectic manifold (M, ω) is exact, i.e. $\omega = -\mathrm{d}\theta$, and the G-action leaves the symplectic potential θ-invariant. The equivariant momentum map will be defined from

$$J_\xi = i(\xi_M)\theta \qquad \forall\ \xi \in \mathfrak{g}. \tag{3.1}$$

Note that, even in this situation, the reduced phase spaces M_μ need not be exact symplectic manifolds.

The periodic orbits of a Hamiltonian system defined by H_t on (M, ω) can be identified with critical points of the action functional (1.1) defined in section 1.

We shall consider the space of smooth free loops $\mathcal{L}_\tau(M)$ on M defined by (1.2). This space can be completed and endowed with the structure of a Hilbert manifold. In what follows we will consider the space of loops $u: \mathbb{R} \to M$, $u(t+\tau) = u(t)$ of Sobolev class 1, that will be denoted by $\Lambda_\tau(M)$ [12]. If u denotes a loop on M, the tangent space at u of $\Lambda_\tau(M)$ can be identified with the Hilbert space of sections of $u^*(TM)$ of Sobolev class 1. It is well known [15] that $\Lambda_\tau(M)$ carries a weak symplectic structure Ω defined as follows:

$$\Omega_u(U, V) = \frac{1}{\tau}\int_0^\tau \omega_{u(t)}(U(t), V(t))\ \mathrm{d}t \qquad \forall\ U, V \in T_u\Lambda_\tau(M). \tag{3.2}$$

If (M, ω) is exact, then $(\Lambda_\tau(M), \Omega)$ is exact, and a symplectic potential is given by

$$\Theta_u(U) = \frac{1}{\tau}\int_0^\tau \theta_{u(t)}(U(t))\ \mathrm{d}t \qquad \forall\ U \in T_u\Lambda_\tau(M). \tag{3.3}$$

The free loop group of G, $\mathcal{L}_\tau(G)$, can be completed to a Hilbert Lie group, that we will denote by $\Lambda_\tau(G)$, by using again as above loops of Sobolev class 1. The geometry of this group has been investigated exhaustively (see for instance [6] and references therein). The action of G on M induces a smooth symplectic action of $\Lambda_\tau(G)$ on $\Lambda_\tau(M)$ by means of

$$\begin{aligned} \tilde{\Phi}:\quad & \Lambda_\tau(G) \times \Lambda_\tau(M) \to \Lambda_\tau(M) \\ & (\tilde{g}, u) \mapsto \tilde{g} \cdot u \end{aligned} \tag{3.4}$$

with

$$\begin{aligned} \tilde{g} \cdot u:\quad & \mathbb{R} \to M \\ & t \mapsto \tilde{g}(t) \cdot u(t). \end{aligned} \tag{3.5}$$

The Lie algebra of the group $\Lambda_\tau(G)$ is $\Lambda_\tau(\mathfrak{g})$, the space of loops of Sobolev class 1 in $\mathfrak{g}$. Choosing a left-invariant metric on G, $\langle\cdot,\cdot\rangle$ we can identify $\mathfrak{g}$ with $\mathfrak{g}^*$, and define an inner product, denoted with the same symbol, in $\mathcal{L}_\tau(\mathfrak{g})$:

$$\langle \tilde{\xi}, \tilde{\zeta} \rangle = \frac{1}{\tau}\int_0^\tau \langle \tilde{\xi}(t), \tilde{\zeta}(t) \rangle\ \mathrm{d}t \qquad \forall\ \tilde{\xi}, \tilde{\zeta} \in \Lambda_\tau(\mathfrak{g}). \tag{3.6}$$

The completion of the loop space $\mathcal{L}_\tau(\mathfrak{g})$ with respect to the L^2-norm defined by (3.6), is the Hilbert space $L^2([0, \tau], \mathfrak{g})$. We shall denote it by $L^2_\tau(\mathfrak{g})$ in what follows. The natural continuous embedding of the Hilbert space $\Lambda_\tau(\mathfrak{g})$ in the Hilbert space $L^2_\tau(\mathfrak{g})$ is compact.

Note that $L^2_\tau(\mathfrak{g}^*) \cong L^2_\tau(\mathfrak{g})^*$, hence we can identify the dual of the Lie algebra of the Hilbert Lie group $\Lambda_\tau(G)$ with a subspace of $L^2_\tau(\mathfrak{g}^*)$ by means of a compact embedding.

A simple computation shows that the infinitesimal generator of the $\Lambda_\tau(G)$-action on $\Lambda_\tau(M)$ associated to $\tilde{\xi}$ is given by the vector field $V_{\tilde{\xi}}$ on $\Lambda_\tau(M)$, which is defined at each point $u \in \Lambda_\tau(M)$ as

$$V_{\tilde{\xi}}(u)(t) = \tilde{\xi}(t)_M(u(t)). \tag{3.7}$$

Proposition 3. The symplectic action of $\Lambda_\tau(G)$ on $\Lambda_\tau(M)$ admits an equivariant smooth momentum map $\mathbb{J}\colon \Lambda_\tau(M) \to L^2_\tau(\mathfrak{g}^*)$ given by

$$\mathbb{J}(u) = J \circ u \qquad \forall\, u \in \Lambda_\tau(M). \tag{3.8}$$

Proof. The smoothness of $\mathbb{J}$ follows from the previous remarks and equivariance follows easily from the definition. Next, note that

$$\mathbb{J}_{\tilde{\xi}}(u) = \langle \tilde{\xi}, \mathbb{J}(u)\rangle = \frac{1}{\tau}\int_0^\tau \langle \tilde{\xi}(t), J(u(t))\rangle \,\mathrm{d}t = \frac{1}{\tau}\int_0^\tau J_{\tilde{\xi}(t)}(u(t))\,\mathrm{d}t. \tag{3.9}$$

Then, for any $U \in T_u\Lambda_\tau(M)$,

$$d_u\mathbb{J}_{\tilde{\xi}}(U) = \frac{1}{\tau}\int_0^\tau d_{u(t)}J_{\tilde{\xi}(t)}(U(t))\,\mathrm{d}t = \frac{1}{\tau}\int_0^\tau \omega_{u(t)}(V_{\tilde{\xi}}(u)(t), U(t))\,\mathrm{d}t = \Omega_u(V_{\tilde{\xi}}(u), U) \tag{3.10}$$

and the conclusion follows. □

From equation (3.8) it follows that $\mathbb{J}^{-1}(\mu) = \Lambda_\tau(J^{-1}(\mu))$, where $\mu \in L^2_\tau(\mathfrak{g}^*)$ denotes the constant loop $\tilde{\mu}(t) = \mu$. The coadjoint orbit in $\Lambda_\tau(\mathfrak{g}^*)$ containing the constant loop μ is the set of loops $\tilde{\mu}(t) = \mathrm{Ad}^*_{\tilde{g}(t)}\,\mu$. This space is diffeomorphic to the space of loops $\Lambda_\tau(\mathcal{O}_\mu)$ on the coadjoint orbit $\mathcal{O}_\mu$. Note that the isotropy group $\Lambda_\tau(G)_\mu$ is precisely $\Lambda_\tau(G_\mu)$. Then, the loop group $\Lambda_\tau(G_\mu)$ acts on $\mathbb{J}^{-1}(\mu)$. Moreover, from equation (3.9), it is clear that if μ is a regular value of J, then it is also a regular value of $\mathbb{J}$.

Proposition 4. Let μ be a regular value of J. The space of loops $\Lambda_\tau(M_\mu)$ on the reduced phase space is symplectically diffeomorphic to the symplectic reduction of $\Lambda_\tau(M)$ with respect to the constant loop μ.

Proof. The map $\Lambda(\pi_\mu)\colon \mathcal{L}_\tau(J^{-1}(\mu)) \to \mathcal{L}_\tau(M_\mu)$ defined in section 2 extends to a unique smooth map from $\mathbb{J}^{-1}(\mu) = \Lambda_\tau(J^{-1}(\mu))$ to $\Lambda_\tau(M_\mu)$ that will be denoted with the same symbol. Because π_μ is a submersion, $\Lambda(\pi_\mu)$ is a submersion too. Besides, if u_1 and u_2 are loops in $\mathbb{J}^{-1}(\mu)$, then $\Lambda(\pi_\mu)(u_1) = \Lambda(\pi_\mu)(u_2)$ if and only if u_1 differs from u_2 in an element on $\mathcal{L}_\tau(G_\mu)$. Then, $\Lambda(\pi_\mu)$ induces a diffeomorphism on the quotient space $\mathbb{J}^{-1}(\mu)/\Lambda_\tau(G_\mu)$. On the other hand, if Ω_μ denotes the symplectic form in $\Lambda_\tau(M_\mu)$, then it is not difficult to check that $\Lambda(\pi_\mu)^*\Omega_\mu = \Lambda(i_\mu)^*\Omega$, where

$$\begin{aligned} \Lambda(i_\mu)\colon \quad & \mathbb{J}^{-1}(\mu) \to \Lambda_\tau(M) \\ & u \mapsto i_\mu \circ u \end{aligned}$$

is the map induced by the canonical inclusion i_μ. The diffeomorphism induced by $\Lambda(\pi_\mu)$ on the quotient is thus a symplectic diffeomorphism. □

In other words, the previous results show that the functor $\Lambda_\tau(\cdot)$ and symplectic reduction commute.

Recall that u is a periodic orbit of the Hamiltonian system defined by H_t if and only if it is a critical point of the smooth functional $\mathcal{A}_H$ defined in the introduction extended to the space $\Lambda_\tau(M)$. Standard regularity arguments show that critical points of $\mathcal{A}_H$ are smooth solutions of Hamilton's equations defined by H_t.

The next proposition gives a characterization of periodic orbits of the reduced Hamiltonian system.

Proposition 5. The restriction $f_\mu = \mathcal{A}_H|_{\mathbb{J}^{-1}(\mu)}$ is invariant under the connected component of the identity, $(\Lambda_\tau(G_\mu))_0$, of the group $\Lambda_\tau(G_\mu)$. Moreover $\mathrm{d}f_\mu$ is invariant under $\Lambda_\tau(G_\mu)$, and it defines a 1-form on $\Lambda_\tau(M_\mu)$ whose zeroes are the periodic orbits of the reduced Hamiltonian system.

Proof. Note first that, for all $u \in \mathbb{J}^{-1}(\mu)$, $U \in T_u(\mathbb{J}^{-1}(\mu))$:

$$
\begin{aligned}
d_u f_\mu(U) &= \frac{1}{\tau}\int_0^\tau \omega_{u(t)}\left(\frac{\mathrm{d}}{\mathrm{d}t}u(t) - X_{H_t}(u(t)), U(t)\right)\,\mathrm{d}t \\
&= \frac{1}{\tau}\int_0^\tau (\pi_\mu^*\omega_\mu)_{u(t)}\left(\frac{\mathrm{d}}{\mathrm{d}t}u(t) - X_{H_t}(u(t)), U(t)\right)\,\mathrm{d}t = \\
&= \frac{1}{\tau}\int_0^\tau (\omega_\mu)_{(\pi_\mu\circ u)(t)}\left(\frac{\mathrm{d}}{\mathrm{d}t}(\pi_\mu\circ u)(t) - X_{h_t}((\pi_\mu\circ u)(t)), T_{u(t)}\pi_\mu U(t)\right)\,\mathrm{d}t.
\end{aligned} \tag{3.11}
$$

The invariance under $\Lambda_\tau(G_\mu)$ of $\mathrm{d}f_\mu$ and the statement about the one-form on $\Lambda_\tau(M_\mu)$ follow easily from this expression. In order to prove the invariance of f_μ under $(\mathcal{L}_\tau(G_\mu))_0$, it is enough to check that the Lie derivative of f_μ in the direction of $V_{\tilde\xi}$ vanishes for every $\tilde\xi \in \Lambda_\tau(\mathfrak{g}_\mu)$, i.e. $d_u f_\mu(V_{\tilde\xi}(u)) = 0$, $\forall\ \tilde\xi \in \Lambda_\tau(\mathfrak{g}_\mu)$. But this also follows from equation (3.11), because $V_{\tilde\xi}(u)(t)$ is tangent to the orbit $G_\mu \cdot u(t)$ for each t. □

Note that f_μ is not invariant under $\Lambda_\tau(G_\mu)$. In fact, it is readily seen that, for any $\tilde g \in \Lambda_\tau(G_\mu)$

$$
(\tilde\Phi_{\tilde g}^* f_\mu)(u) = f_\mu(u) + \frac{1}{\tau}\int_0^\tau J_{\tilde\zeta(t)}(u(t))\,\mathrm{d}t = f_\mu(u) + \frac{1}{\tau}\int_0^\tau \langle\tilde\zeta(t), \mu\rangle\,\mathrm{d}t \tag{3.12}
$$

where $\tilde\zeta \in \Lambda_\tau(\mathfrak{g}_\mu)$ is defined as

$$
\tilde\zeta(t) = TL_{\tilde g(t)^{-1}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\tilde g(t)\right) \tag{3.13}
$$

with L denoting left translation in G_μ.

The map f_μ defines a map in the quotient space $\widehat{\Lambda_\tau(M_\mu)} = \mathbb{J}^{-1}(\mu)/(\Lambda_\tau(G_\mu))_0$, which is a principal fibre bundle over $\Lambda_\tau(M_\mu)$ with structural group $\pi_1(G_\mu) \cong \pi_0(\mathcal{L}_\tau(G_\mu))$, and hence a multivalued functional $\mathcal{L}_\tau(M_\mu) \to \mathbb{R}$ with a well defined variation given by the closed one-form above.

Using the results of section 2, we can give an alternative characterization of periodic orbits of the reduced system.

If u_μ is a periodic orbit of a reduced Hamiltonian system defined by h_t, then the periodic orbits associated to it in proposition 2 will be the critical points of the family of action functionals

$$\mathcal{A}_{H,\xi}(u) = \frac{1}{\tau}\int_u \theta - \frac{1}{\tau}\int_0^\tau H_t(u(t),t)\,\mathrm{d}t - \frac{1}{\tau}\int_0^\tau J_\xi(u(t))\,\mathrm{d}t \qquad \xi \in \mathfrak{g} \tag{3.14}$$

with $\xi \in \mathfrak{g}_\mu$ and lying on $\mathbb{J}^{-1}(\mu)$.

Let us introduce the averaged momentum map $\mathcal{J}$, defined by

$$\mathcal{J}(u) = \frac{1}{\tau}\int_0^\tau J(u(t))\,\mathrm{d}t. \tag{3.15}$$

The constant loops $\tilde{g}(t) = g$ form a subgroup of $\Lambda_\tau(G)$ isomorphic to G, and $\mathcal{J}\colon \Lambda_\tau(M) \to \mathfrak{g}^*$ is precisely the momentum map for the corresponding symplectic G-action on $\Lambda_\tau(M)$.

Now, if u is a critical point of $\mathcal{A}_{H,\xi}$, with $\xi \in \mathfrak{g}$, then from

$$\frac{\mathrm{d}}{\mathrm{d}t}u(t) = X_{H_t}(u(t)) + \xi_M(u(t)) \tag{3.16}$$

one obtains

$$\frac{\mathrm{d}}{\mathrm{d}t}(J\circ u)(t) = -\,\mathrm{ad}^*_\xi((J\circ u)(t)) \tag{3.17}$$

and hence

$$J(u(t)) = \mathrm{Ad}^*_{\exp(-t\xi)}\, J(u(0)) \qquad \forall\ t \in \mathbb{R}. \tag{3.18}$$

The condition $u \in \mathbb{J}^{-1}(\mu)$ implies $\xi \in \mathfrak{g}_\mu$. Then, the periodic orbits associated to a given periodic orbit in the reduced system are the critical points of all the action functionals (3.14) lying on $\mathbb{J}^{-1}(\mu) \subset \mathcal{J}^{-1}(\mu)$, which may be seen as critical points of the restriction

$$\hat{f}_\mu = \mathcal{A}_H|_{\mathcal{J}^{-1}(\mu)} \tag{3.19}$$

if one considers the components of $\xi \in \mathfrak{g}$ as Lagrange multipliers.

Thus we have proved:

Proposition 6. The periodic curves in $\widehat{\mathcal{P}}^\tau_H$ are the critical points of

$$\hat{f}_\mu = \mathcal{A}_H|_{\mathcal{J}^{-1}(\mu)} \tag{3.20}$$

lying on $\mathbb{J}^{-1}(\mu)$.

To each periodic orbit of the Hamiltonian system defined by h_t there will correspond a critical subset of $\hat{f}_\mu$ of the form given in proposition 2.

Note that, if G is Abelian, the coadjoint action on $\mathfrak{g}^*$ is trivial, and the critical points of $\hat{f}_\mu$ all lie on $\mathbb{J}^{-1}(\mu)$, i.e. on critical loops, the condition $\mathcal{J}(u) = \mu$ implies the pointwise condition $J(u(t)) = \mu,\ \forall\ t \in \mathbb{R}$.

Now, let $u_1, u_2 \in \mathbb{J}^{-1}(\mu)$ be any two critical points of $\hat{f}_\mu$ projecting on the same periodic orbit in M_μ. Then

$$u_2(t) = \Phi_{g(\exp t\xi)(\exp(-t\zeta))}u_1(t) \tag{3.21}$$

with $\xi, \zeta \in \mathfrak{g}_\mu$, $\exp\tau\xi = \exp\tau\zeta$ and $g \in G_\mu$.

Hence the points in $\widehat{\Lambda}(\pi_\mu)^{-1}(u_\mu)$, for a given $u_\mu \in \mathcal{P}^\tau_h$, are critical points of f_μ belonging to the same $\mathcal{L}_\tau(G_\mu)$-orbit in $\mathbb{J}^{-1}(\mu)$. The connected components of the orbit in which they fall are determined by the homotopy classes of the curves $t \mapsto (\exp t\xi)(\exp(-t\zeta))$ in $\pi_1(G_\mu) \cong \pi_0(\mathcal{L}_\tau(G_\mu))$ (recall that $\Lambda_\tau(G_\mu)$ and $\mathcal{L}_\tau(G_\mu)$ are homotopically equivalent). Let $u_1, u_2 \in \mathbb{J}^{-1}(\mu)$ as above, then a short computation shows that

$$\hat{f}_\mu(u_1) - \hat{f}_\mu(u_2) = \langle \xi - \zeta, \mu \rangle. \tag{3.22}$$

Proposition 7. For each periodic orbit $u_\mu \in \mathcal{P}_h^\tau$ there is one orbit of $\mathcal{L}_\tau(G_\mu)$ of critical points of f_μ on $\mathbb{J}^{-1}(\mu)$. Moreover if u_1, u_2 are two such critical points, there will exist $\tilde{\eta}$ in $\mathcal{L}_\tau(\mathfrak{g}_\mu)$ such that their critical values are related by

$$f_\mu(u_1) - f_\mu(u_2) = \langle \tilde{\eta}, \mu \rangle. \tag{3.23}$$

Proof. From proposition 5 we know that $\mathrm{d}f_\mu$ is invariant with respect to the action of $\Lambda_\tau(G_\mu)$. Then, the critical points of f_μ will consist of $\mathcal{L}_\tau(G_\mu)$ orbits on $\mathbb{J}^{-1}(\mu)$. Note that as in (3.21), if u_1, $u_2 \in \mathbb{J}^{-1}(\mu)$ are two critical points of f_μ, then

$$u_2(t) = \Phi_{g(\exp \xi(t))(\exp(-\zeta(t)))} u_1(t) \tag{3.24}$$

with $\xi(t), \zeta(t) \in \mathcal{L}_\tau(\mathfrak{g}_\mu)$, $\exp \xi(\tau) = \exp \zeta(\tau)$ and $g \in G_\mu$. Defining $\tilde{\eta} = \xi - \zeta$ and because of (3.12), we obtain the desired formula. □

Note that, by the remarks above, the number of critical points of f_μ on $\widehat{\Lambda_\tau(M_\mu)}$ are always $\#\pi_0(\mathcal{L}_\tau(G_\mu)) = \#\pi_1(G_\mu)$, the number of connected components of the $\mathcal{L}_\tau(G_\mu)$-orbit, but the number of critical points of $\hat{f}_\mu$ could be strictly lower.

Again, when G is Abelian, the situation is far simpler and the family of critical values corresponding to a periodic orbit in the reduced system is parametrized by the discrete set $\langle \exp^{-1}(e), \mu \rangle$.

4. Toric actions and Hamilton's principle

In this section we will apply the previous results to symplectic reduction by an Abelian compact group. In addition we will show how any symplectic manifold (M, ω) with ω having finite integral rank can be realized as the Marsden–Weinstein reduction of an exact symplectic manifold with respect to a torus action, so that Hamilton's principle can be applied to reduced Hamiltonian systems. The finite integral rank condition is not very restrictive and it is satisfied, in particular, if the manifold M is of finite type.

Let (M, ω) be a symplectic manifold with ω of finite integral rank, i.e. such that $[\omega] \in H^2(M, \mathbb{R})$ lies in $H^2(M, \mathbb{Z}) \otimes \mathbb{R}$. Then, there exist integral closed 2-forms $c_1, \ldots, c_N$ and non-zero real numbers $a_1, \ldots, a_N$ such that $\omega = \sum_{i=1}^N a_i c_i$.

For each $i = 1, \ldots, N$ there is a principal S^1-bundle $P_i \xrightarrow{\pi_i} M$ with connection A_i whose curvature satisfies $\mathrm{d}A_i = \pi_i^* c_i$. The fibre product $P \xrightarrow{\pi} M$ of the P_i is a principal $\mathbf{T}^N$-bundle.

In what follows, we shall identify the Lie algebra $\mathfrak{t}^N$ of $\mathbf{T}^N$ with $\mathbb{R}^N$. If p_i denotes the projection $P \to P_i$, $i = 1, \ldots, N$, then $A = (p_1^* A_1, \ldots, p_N^* A_N)$ defines a connection in P with curvature $\mathrm{d}A = (\pi^* c_1, \ldots, \pi^* c_N)$.

Let us consider now the closed 2-form $\omega_P = \pi^* \omega$ in P. From

$$\omega_P = \pi^* \left(\sum_{i=1}^N a_i c_i \right) = \sum_{i=1}^N a_i \pi^* c_i = \sum_{i=1}^N a_i p_i^* \mathrm{d}A_i \tag{4.1}$$

one obtains $\omega_P = -\mathrm{d}\theta_P$ with $\theta_P = -\sum_{i=1}^N a_i p_i^* A_i = -\langle A, a \rangle$, where $a = (a_1, \ldots, a_N) \in (\mathfrak{t}^N)^* \cong \mathbb{R}^N$.

The manifold (P, ω_P) is a presymplectic manifold with characteristic bundle $\ker\omega_P = V(P)$, the vertical subbundle of TP, which is a trivial bundle as well as its dual

$$\ker\omega_P \cong P \times \mathbb{R}^N \qquad \text{and} \qquad (\ker\omega_P)^* \cong P \times \mathbb{R}^N. \tag{4.2}$$

The connection A allows to define an exact symplectic form Ω in a neighbourhood $\mathcal{U}$ of the zero section $P \times \{\mathbf{0}\}$ in $P \times \mathbb{R}^N$ such that the map

$$\begin{aligned} \iota\colon P &\to \mathcal{U} \\ p &\mapsto (p, \mathbf{0}) \end{aligned} \tag{4.3}$$

is a coisotropic embedding (see [7, 8]).

Let $\mathrm{pr}_1\colon P \times \mathbb{R}^N \to P$ and $\mathrm{pr}_2\colon P \times \mathbb{R}^N \to \mathbb{R}^N$ be the natural projections and consider the following 1-form in $P \times \mathbb{R}^N$:

$$\Theta = \mathrm{pr}_1^*\theta_P - \langle \mathrm{pr}_1^* A, \mathrm{pr}_2\rangle = -\langle \mathrm{pr}_1^* A, \mathrm{pr}_2 + a\rangle \tag{4.4}$$

i.e. for $(p, \mu) \in P \times \mathbb{R}^N$ we have

$$\Theta_{(p,\mu)}(U) = -\langle A_p(U_1), \mu + a\rangle \tag{4.5}$$

for each $U = (U_1, U_2)$ in $T_{(p,\mu)}(P \times \mathbb{R}^N) \cong T_pP \times T_\mu\mathbb{R}^N \cong T_pP \times \mathbb{R}^N$.

The 2-form $\Omega = -\mathrm{d}\Theta$ is non-degenerate at the points of $P \times \{\mathbf{0}\}$ and hence in some neighbourhood $\mathcal{U}$ of $P \times \{\mathbf{0}\}$. This is easily seen from the explicit expression for Ω:

$$\Omega_{(p,\mu)}(U, V) = (\pi^*\omega)_p(U_1, V_1) + \langle(\mathrm{d}A)_p(U_1, V_1), \mu\rangle + \langle A_p(V_1), U_2\rangle - \langle A_p(U_1), V_2\rangle \tag{4.6}$$

for $U = (U_1, U_2)$, $V = (V_1, V_2) \in T_{(p,\mu)}(P \times \mathbb{R}^N)$. If M is compact, then the neighbourhood $\mathcal{U}$ can be chosen of the form $\mathcal{U} = P \times \mathcal{V}$, for some neighbourhood $\mathcal{V}$ of $\mathbf{0}$ in $\mathbb{R}^N$.

The diagonal action of $\mathbf{T}^N$ on $P \times \mathbb{R}^N$

$$g \cdot (p, \mu) = (p \cdot g^{-1}, \mathrm{Ad}^*_{g^{-1}}\,\mu) = (p \cdot g^{-1}, \mu) \qquad \forall\ g \in \mathbf{T}^N \quad \forall\ (p, \mu) \in P \times \mathbb{R}^N \tag{4.7}$$

leaves the symplectic potential Θ invariant. Moreover, the neighbourhood $\mathcal{U}$ above can be chosen $\mathbf{T}^N$-invariant, so that we have a symplectic action of $\mathbf{T}^N$ on $(\mathcal{U}, \Omega)$ with an equivariant momentum map J defined by the Hamiltonians

$$J_\xi = i(\xi_{\mathcal{U}})\Theta \qquad \xi \in \mathbb{R}^N. \tag{4.8}$$

The infinitesimal generators of the action on $\mathcal{U}$ are given by

$$\xi_{\mathcal{U}}(p, \mu) = (\xi_P(p), -\mathrm{ad}^*_{\xi(\mu)}) = (\xi_P(p), \mathbf{0}) \tag{4.9}$$

and the corresponding Hamiltonians by

$$J_\xi(p, \mu) = -\langle \xi, \mu + a\rangle. \tag{4.10}$$

The equivariant momentum map for the symplectic action of $\mathbf{T}^N$ on $\mathcal{U}$ is thus $J = -(\mathrm{pr}_2 + a)$ and the Marsden–Weinstein reduction of $(\mathcal{U}, \Omega)$ with respect to $-a \in \mathbb{R}^N$ is isomorphic to (M, ω).

Now, let $h : M \times \mathbb{R} \to \mathbb{R}$ be a Hamiltonian on M, 1-periodic in time, and consider any $H : \mathcal{U} \times \mathbb{R} \to \mathbb{R}$, 1-periodic in time and such that each H_t is an invariant extension to $\mathcal{U}$ of the function $P \times \{\mathbf{0}\} \to \mathbb{R}$ defined by $(p, \mathbf{0}) \mapsto h_t(\pi(t))$. For example, we can take

$$\begin{aligned} H:\ & \mathcal{U} \times \mathbb{R} \to \mathbb{R} \\ & ((p, \mu), t) \mapsto h(\pi(p), t). \end{aligned} \tag{4.11}$$

Note that, with this particular choice of H, the associated Hamiltonian vector field X_{H_t} satisfies

$$X_{H_t}(p, \mathbf{0}) = (X_t(p), \mathbf{0}) \tag{4.12}$$

where X_t is the horizontal lifting of X_{h_t} to P using the connection A.

The relative periodic orbits introduced in section 2 are then curves $(\sigma_p, \mathbf{0})$, with σ_p being the horizontal lifting of a periodic orbit in (M, ω), and the holonomy defined in (2.11) is precisely the holonomy of the connection A along the path σ_p.

The variational characterization of periodic orbits in the reduced Hamiltonian system will be given as follows. Let us consider the action functional $\mathcal{A}_H$ on $\Lambda_\tau(\mathcal{U}) \subset \Lambda_\tau(P \times \mathbb{R}^N) \cong \Lambda_\tau(P)\hat{\otimes}\Lambda_\tau(\mathbb{R}^N)$:

$$\mathcal{A}_H(u) = \frac{1}{\tau}\int_u \Theta - \frac{1}{\tau}\int_0^\tau H_t(u(t))\,\mathrm{d}t \tag{4.13}$$

and the averaged momentum map

$$\begin{aligned} \mathcal{J}:\quad & \Lambda_\tau(\mathcal{U}) \to \mathbb{R}^N \\ & u = (u_1, u_2) \mapsto -a - \frac{1}{\tau}\int_0^\tau u_2(t)\,\mathrm{d}t. \end{aligned} \tag{4.14}$$

By the results of the previous sections, particularized to the case of a free torus action, to each periodic orbit with period $\tau \in \mathbb{N}$ of the Hamiltonian system defined by h on (M, ω) there corresponds a lattice, bijective to $\mathbb{Z}^N$, of critical $\mathbf{T}^N$-orbits of the restriction of $\mathcal{A}_H$ to the submanifold $\mathcal{J}^{-1}(-a) = \{(u_1, u_2) \in \mathcal{L}_\tau(\mathcal{U}) \mid \int_0^\tau u_2(t)\,\mathrm{d}t = 0\}$.

Moreover, the corresponding set of critical values is parametrized by $\langle (2\pi\mathbb{Z})^N, a\rangle$. Since a comes from the decomposition $\omega = \sum_{i=1}^N a_i c_i$, its components a_i can be taken independent over $\mathbb{Z}$, and hence the set of critical values corresponding to a periodic orbit is also bijective to $\mathbb{Z}^N$, i.e. each critical $\mathbf{T}^N$-orbit contributes with a critical value.

Indeed, the critical points of $\hat{f} = \mathcal{A}_H|_{\mathcal{J}^{-1}(-a)}$ all lie on $\mathcal{L}_\tau(P \times \{\mathbf{0}\})$ and since the map

$$\begin{aligned} &\exp^{-1}(e) \to \pi_1(\mathbf{T}^N) \\ &\xi \mapsto [\exp t\xi] \end{aligned}$$

is surjective, there are critical points of $\hat{f}$ on each connected component of a $\mathcal{L}_\tau(\mathbf{T}^N)$-orbit in $\Lambda_\tau(P \times \{\mathbf{0}\})$. In other words, each periodic orbit of the reduced system gives rise to a critical $\pi_1(\mathbf{T}^N)$-orbit in the space $\widehat{\Lambda_\tau(M)}$ introduced in section 3.

It is relevant to point out here that this method is close in spirit to the universal lifting of Arnold's conjecture to $\mathbb{R}^{2N}$ discussed in [10, 11] and which is based upon the universal symplectic unreduction of symplectic manifolds [9].

Non-Abelian situations, like Hamiltonian systems on coadjoint orbits of compact Lie groups obtained by symplectic reduction of cotangent groups, and their applications, will be discussed elsewhere.

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